JOURNAL OF
PURE AND
APPLIED ALGEBRA

ELSEVIER

Journal of Pure and Applied Algebra 149 (2000) 15–45

www.elsevier.com/locate/jpaa

Axiomatic etal maps and a theory of spectrum

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Received 10 February 1996; received in revised form 14 September 1998
Communicated by F.W. Lawvere

Abstract

We consider and develop the axioms introduced by A. Joyal that define an abstract notion of an *etal class* $\mathbf{A}$ of arrows in a Grothendieck topos $\mathbf{E}$. The axioms are intended to be sufficient in the sense that the category of objects etal over the terminal object (and etal maps between them) should be a topos, the 'Etal' Topos $\mathbf{Et}$. It can be shown that $\mathbf{Et} \subset \mathbf{E}$ is a full subcategory, closed under all colimits and finite limits, and so it is almost a topos. However, the problem of the existence of generators for $\mathbf{Et}$ and the existence of a right adjoint for the inclusion making it the inverse image of a geometric morphism $\mathbf{E} \to \mathbf{Et}$ is still open.

We introduce an additional axiom that we call the *etal topology condition* or ETC, and for a topos $\mathbf{E}$ equipped with such a class, we develop a general construction of *germs* which yields a new point associated to any given point $\mathbf{F} \to \mathbf{E}$.

A particular case of this furnishes a right adjoint for the inclusion $\mathbf{Et} \subset \mathbf{E}$, and it follows that in this case $\mathbf{Et}$ is a subtopos of $\mathbf{E}$, the center of a *local* geometric morphism of topoi. $\mathbf{E} \to \mathbf{Et}$.

Also, we introduce a new general theory of Spectrum where etal classes take care of the role assigned to the admissible morphisms, and prove a general theorem of existence based in the construction of germs. This theorem includes all the known results in the theory of Cole's spectrum. This theory is more general since it is associated to any geometric morphism rather than only to the inclusion of subtopoi, and conceptually it is independent from the notion of geometric theory. © 2000 Elsevier Science B.V. All rights reserved.

MSC: 18B25; 18F99

0. Introduction

An etal map is a map that *in a sense* is locally surjective and locally injective. We consider axioms for a class $\mathbf{A}$ of arrows in a Grothendieck topos $\mathbf{E}$. These axioms define the notion of an *etal class*, and were introduced by Joyal in several lectures given in the late 1970s. They are properties for classes of maps explicitly considered

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PII: S0022-4049(98)00161-3

in several contexts in the SGA volumes. See for example Exposé VI, SGA4, and for the particular context in this paper, the "Médaille en Chocolat" exercise, 4.10.6, Exposé IV SGA4. They are divided in two groups (see Section 1). The first group make sense in any category with finite limits, and the second group relates to a notion of covers or epimorphism. Joyal explicitly uses the axioms as a syntactic method to generate etal maps in the topos out of a basic set of (etal) maps (typically in a site of definition). These axioms are intended to be sufficient in the sense that the category of objects etale over the terminal object (and etal maps between them) should be a topos, the 'Etal' Topos, $\mathbf{Et} \subset \mathbf{E}$. This is so in all the examples, and it can be proved under additional assumptions. To show this in all generality is the *principal problem* of the theory, which is still open.

The axioms suffice to show that $\mathbf{Et} \subset \mathbf{E}$ is a full subcategory, closed under all colimits and finite limits, and so it is almost a topos. However, we still cannot prove the existence of generators for **Et**, neither that of a right adjoint for the inclusion $\mathbf{Et} \subset \mathbf{E}$, making it the inverse image of a geometric morphism $\mathbf{E} \to \mathbf{Et}$, so that **E** becomes an **Et**-Topos.

Joyal also stated a completeness theorem for the axioms. An arrow $h : X \to Y$ in **E** and a morphism between points $\theta : p \to q : \mathbf{F} \to \mathbf{E}$ determine a commutative square

$$\begin{tikzcd} p^*(X) \arrow[r, "p^*(h)"] \arrow[d, "\theta X"] & p^*(Y) \arrow[d, "\theta Y"] \\ q^*(X) \arrow[r, "q^*(h)"] & q^*(Y) \end{tikzcd}$$

When this square is a pull-back, θ is said to be an *infinitesimal extension* with respect to h, and h to be *etal* with respect to θ.

Completeness means that given any set of arrows $\underline{A}$, if an arrow is etal with respect to all infinitesimal extensions determined by $\underline{A}$, then it is generated by the arrows in $\underline{A}$ via the axioms.

Completeness in this sense implies an affirmative solution to the principal problem. Here we show this in Theorem 3.2 (see also [3, 7]). However, it is false in all generality, as it follows from [7, Remark 6.7].

In the central part of the paper, we develop a general construction of *germs* in a topos utilizing an abstract class of etal maps. This construction yields, depending on the context, the classical construction of germs of continuous functions, the Godement construction of the etal space, the localization of a ring in a prime ideal, etc.

We introduce a condition that we call the *etal topology condition* or ETC, which relates the etal class with the epimorphisms, or, more concretely, the etal arrows in the site with the coverings. Then we prove the principal result in the paper, Theorem 5.2, which says that under condition ETC, given any point $s : \mathbf{F} \to \mathbf{E}$, the category of infinitesimal extensions below s has a terminal object. This terminal object is the output of the construction of germs.

In Theorem 7.3 we show that under condition ETC the completeness problem has an affirmative answer, since completeness follows from Theorem 5.2. We originally presented this proof at the 1987 Louvain la Neuve Conference (see [3]). Joyal had also suggested at the time a line of proof based on a concrete construction of the classifying topos for infinitesimal extensions, and he gave an explicit description of a site for this topos. Using this, in collaboration with Moerdijk he proved in 1990 a completeness theorem for classes which satisfy an additional axiom (see [7]). Since condition ETC implies this axiom, our result follows. However our proof is completely different, as it is based in the supply of infinitesimal extensions provided by Theorem 5.2. This theorem implies that there are enough infinitesimal extensions, which is, in another sense, also the meaning of the completeness theorem.

Theorem 5.2 implies directly that the principal problem has a strong affirmative answer. Explicitly, the inclusion $\mathbf{Et} \subset \mathbf{E}$ is not only the inverse image of a geometric morphism, but this morphism is a *local* morphism of topoi. The right image is given by the terminal object in the category of infinitesimal extensions below the identity (or *generic*) point of $\mathbf{E}$. That is, in this case the construction of germs yields the right adjoint to the inclusion $\mathbf{Et} \subset \mathbf{E}$.

We exploit this result to produce a general construction of *Spectrum* inspired by Hakim's work, that includes all examples and results of Cole's Spectrum. We introduce a new general theory of Spectrum, where etal classes take care of the role assigned to the admissible morphisms in Cole's spectrum. This theory is more general since it is associated to any geometric morphism of topoi rather than only to the inclusion of subtopoi. Conceptually, it frees the notion of spectrum from the notion of geometric theory. Etal classes are used to define admissible morphisms, rather than to take these as the primitive notion. Attention is focused on the topos rather than on its points. In Theorem 9.2 we develop the abstract construction of the spectrum (based in the construction of germs of Theorem 5.2), and in Theorem 9.3 we give a proof for its existence . This construction is more akin to the geometric intuition behind the classical examples of spectra, and the existence theorem includes all the known results.

This is a rewritten version of our preprint [3]. It differs in several ways. A mistake in the proof of Theorem 5.2 is corrected, and all consequences of our mistaken belief are eliminated. The part concerning spectrum theory has been enlarged and entirely rewritten, and to do this properly we had to add an appendix with some constructions which are not found in the literature.

We thank A. Kock, A. Joyal, J. Penon and especially B. Lawvere for some fruitful conversations on the subject, and A. Joyal for indicating counterexamples to results in [3].

1. The axioms for a class of etal maps

All topoi are *Grothendieck topoi*. All morphisms of topoi are *Geometric morphisms*. A morphism $p : \mathbf{F} \to \mathbf{E}$ will be called also an $\mathbf{F}$-*point* of $\mathbf{E}$. A map between $\mathbf{F}$-points $\theta : p \to q$ is a natural transformation $\theta : q^* \to p^*$ between the inverse image functors (notice we adopt Grothendieck's convention of reversal of the arrow). A *Model* (*of* **E**) *in* **F** is (by definition) the inverse image of a morphism $\mathbf{F} \to \mathbf{E}$.

1.1. On the concept of etal map. A local homeomorphism between topological spaces $h: X \to Y$ is a continuous function which is *locally surjective*, that is, *open*

$$\forall x \in X,\ \forall U \mid x \in U,\ \exists V \mid h(x) \in V \subset h(U).$$

and *locally injective*, that is

$$\forall x \in X,\ \exists U \mid x \in U \text{ such that } h \mid U \text{ is injective.}$$

(where U, V are open subsets of X and Y, respectively).

This last condition is equivalent to the fact that the diagonal inclusion into the pull-back, $\Delta_h: X \to X \times_Y X$, is also an open map.

These ideas are behind the abstract concept of *etale map*. An etale map is a map that *in a sense* is locally surjective and locally injective. Local injectivity means that the diagonal inclusion into the pull-back is locally surjective. (thus, if h is etal, so is Δ_h).

1.2. The axioms. Joyal proposed a list of axioms for an (abstract) notion of etal maps (he actually considered axioms for open maps, from which he deduced the axioms for etal maps via the equation $h\,\text{etal} = h\,\text{open} + \Delta_h$ open. Given a topos **E**, the axioms for a class **A** of *etal* arrows in **E** are the following:

First Group: (for any category with finite limits)

(A0) $\quad h \in \mathbf{A} \quad \Rightarrow \quad \Delta_h \in \mathbf{A}$,

(A1) $\quad h$ isomorphism $\quad \Rightarrow \quad h \in \mathbf{A}$,

(A2) $\begin{tikzcd} \cdot \arrow[r, "h"'] \arrow[dr, phantom, ""] & \cdot \\ & \end{tikzcd}$ $\quad$ with $h = g \circ f$: $\begin{tikzcd} & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "h"'] & & \cdot \end{tikzcd}$ $\quad f, g \in \mathbf{A} \quad \Rightarrow \quad h \in \mathbf{A}$,

(A3) $\begin{tikzcd} \cdot \arrow[r] \arrow[d, "h"'] \arrow[dr, phantom, "p.b"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$ $\quad g \in \mathbf{A} \quad \Rightarrow \quad h \in \mathbf{A}$.

Second Group: (good epimorphisms and coproducts are necessary)

(A4) $\begin{tikzcd} \cdot \arrow[r, two heads] \arrow[d, "f"'] \arrow[dr, phantom, "p.b"] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, two heads] & \cdot \end{tikzcd}$ $\quad f \in \mathbf{A} \quad \Rightarrow \quad h \in \mathbf{A}$,

(A5) $\begin{tikzcd} & \cdot \arrow[dr, "h"] & \\ \cdot \arrow[ur, "g"] \arrow[rr, "f"'] & & \cdot \end{tikzcd}$ $\quad \Delta_h, f \in \mathbf{A} \quad \Rightarrow \quad h \in \mathbf{A}$,

(A6) $\begin{tikzcd} \cdot \arrow[r, "\lambda_i"] \arrow[dr, "h_i"'] & \Sigma_i \arrow[d, "h"] \\ & \cdot \end{tikzcd}$ $\quad (i \in J \in \mathbf{S}) \quad \forall_i \lambda_i \in \mathbf{A}$,

$\quad \forall_i h_i \in \mathbf{A} \quad \Rightarrow \quad h \in \mathbf{A}$.

Here, if $h : X \to Y$, "Δ_h" denotes the diagonal inclusion $\Delta_h : X \to X \times_Y X$, "$p.b$" means that the square is a pull-back, "$\twoheadrightarrow$" indicates "epimorphism", and "$\sum_i$" a coproduct of objects in $\mathbf{E}$ (indexed by J in $\mathbf{S}$), with inclusions "λ_i".

Axioms A1–A4 were considered by Grothendieck in the "Médaille en Chocolat" exercise, SGA 4 IV 4.10.6, and are some of the known properties of etal maps between Schemes. It is Joyal's contribution that he could add axioms A5 and A6 to generate (together with A4) etal maps in the topos out of etal maps in the site. Actually, in place of A5 he had the statement in Proposition 2.5 below as an axiom. This statement is equivalent to A5, for the other implication see [7, Proposition 1.10]. The following is easy (see [3, 7]).

1.3. Definition–Proposition. Intersection of etal classes is an etal class. The whole topos is an etal class. Thus, any collection $\underline{A}$ of arrows is contained in a smallest etal class. This class is generated by the arrows in $\underline{A}$ via the axioms. We shall denote it $\mathrm{cl}(\underline{A})$. Thus, by definition, given any etal class $\mathbf{A}$, we have

$$\underline{A} \subset \mathbf{A} \Leftrightarrow \mathrm{cl}(\underline{A}) \subset \mathbf{A}.$$

In particular, there is the smallest etal class, generated by the isomorphisms. In the base topos $\mathbf{S}$ there are no proper etal classes.

1.4. The etal class of pull-back squares. This example plays a central role in the theory. For technical reasons (see Proposition 1.4.1 below) we need to consider also a weaker notion. Recall that a commutative square is a *quasipull-back* if the induced morphism into the pull-back is an epimorphism.

This notion has independent interest, and furnishes a basic example for the axioms for a *class of open maps*. However, it shall not play any role in this paper other than its technical use in the proof of Proposition A.2.4.

1.4.1. Definition–Proposition. Given any topos $\mathbf{E}$, consider the topos of contravariant $\mathbf{E}$-valued functors defined on $\mathbf{2} = \{0 \to 1\}$. Its objects are arrows (in $\mathbf{E}$), and its arrows are commutative squares. Let $\mathbf{P}$ and $\mathbf{Q}$ be the classes of pull-back and quasipull-back squares, respectively. Then

The class $\mathbf{P}$ is an etal class in $\mathbf{E}^{\mathbf{2}op}$. Moreover, given any arrow h in $\mathbf{E}^{\mathbf{2}op}$, we have

$$h \in \mathbf{P} \quad \Leftrightarrow \quad h \in \mathbf{Q} \text{ and } \Delta_h \in \mathbf{Q}.$$

Proof. We leave the proof as an exercise. See also [7, Remark 1.11]. □

When $\mathbf{E} = \mathbf{S}$, $\mathbf{S}^{\mathbf{2}op}$ is the topos of sheaves on the Sierpinski space. In this case, pull-back squares are the smallest etal class and the only proper class in this topos. This can be seen as follows: the only proper subobject of 1 is etal or is not etal. In the first case the class is the whole topos, and in the second it is the class of pull-back squares.

2. Elementary consequences of the axioms

We consider a topos **E** furnished with an etal class **A**. An object H in **E** is *etale* if the (unique) arrow $H \to 1$ is in **A**. The aim of this section is to show that etal objects together with etal arrows form a full subcategory **Et** of **E** closed under all finite limits and **S**-indexed colimits. This means that **Et** will be a Grothendiek topos if we can show that it has generators. It is not known how to do this in all generality, and it still remains an open problem in the theory. Also open is the (weaker) question as to whether the inclusion (of **Et** into **E**) has always a right adjoint. These problems are related to the completeness of the axioms in a sense that we will make precise in Section 7.

2.1. Definition. Given any object Z in **E**, the category of objects *etal over* Z, denoted $\mathbf{Et}/_Z$, is the subcategory of $\mathbf{E}/_Z$ whose objects are etal maps $H \to Z$, and whose arrows are etal maps making the usual triangle commutative. When $Z = 1$, we just denote $\mathbf{Et} \subset \mathbf{E}$. Thus, while **A** is the class of etal maps between all objects, **Et** is the class of all maps between etal objects. (as we shall see, **Et** is full).

We examine first what follows from only the first group of axioms, and which will hold in any category **E** with finite limits (see [3;7;11, exposé VI]).

2.2. Proposition.
(a) *For any commutative triangle*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$

if $\Delta_g, h \in \mathbf{A}$ *then* $f \in \mathbf{A}$.

(b) *Given a diagram*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "h"'] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "s"] & Q \arrow[dl, "t"] \\ & Z & \end{tikzcd}$$

where the triangle commutes and i *is an equalizer*, *then*

if $\Delta_t \in \mathbf{A}$ *then* $i \in \mathbf{A}$.

We remark that only axioms $A1$, $A2$ and $A3$ are needed in the proof of this proposition.

Proof. (a) Consider the following commutative diagram:

$$\begin{tikzcd}
 & X \arrow[r, "f"] \arrow[dl, "f"'] \arrow[d, "\Gamma_f"] & Y \arrow[d, "\Delta_g"] \\
Y \arrow[d, "g"] & X \times_Z Y \arrow[l, "\pi_2"'] \arrow[r, "f \times Y"] \arrow[d, "\pi_1"] & X \times_Z Y \\
Z & X \arrow[l, "h"'] &
\end{tikzcd}$$

where the two squares are pull-backs, π_1, π_2 are the projections and $\Gamma_f = (\mathrm{id}, f)$ is the graph of f. Then, π_2 and Γ_f are in $\mathbf{A}$ by $A3$. Thus, f is in $\mathbf{A}$ by $A2$. □

Proof. (b) Apply $A3$ in the following pull-back diagram:

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "i"] & Y \arrow[d, "\Delta_t"] \\
X \arrow[r, "(f,g)"] & Y \times_Z Y
\end{tikzcd} \qquad \square$$

2.3. Proposition. *For any object Z, $\mathbf{Et}/_Z \subset \mathbf{E}/_Z$ is a full subcategory closed under all finite limits.*

Proof. This is a consequence of $A0$ together with Proposition 2.2. From Proposition 2.2(a) it immediately follows that $\mathbf{Et}/_Z$ is a full subcategory. Then, by $A3$ products taken in $\mathbf{E}/_Z$ are in $\mathbf{Et}/_Z$, and from Proposition 2.2(b), $A0$ and $A2$ it follows that equalizers taken in $\mathbf{E}/_Z$ are in $\mathbf{Et}/_Z$. □

2.4. Remark. Notice that $A0$ is not only necessary for equalizers, but also in the case of products. Even if by $A3$ the product is in $\mathbf{Et}/_Z$, it will not have the universal property unless the subcategory is full. On the other hand, since the diagonal is an equalizer of the two projections, in the presence of axioms $A1$–$A3$ axiom $A0$ is actually equivalent to the fact that $\mathbf{Et}/_Z$ is closed under finite limits. Thus, we cannot have this unless $A0$ holds.

We pass now to prove some consequences of the whole set of axioms in a topos (see [3, 7]).

2.5. Proposition. *For any epimorphism f in a triangle as follows*:

$$\begin{tikzcd}
X \arrow[rr, two heads, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\
 & Z &
\end{tikzcd}$$

if $g, f \in \mathbf{A}$ then $h \in \mathbf{A}$.

Proof. Consider the diagram

$$\begin{tikzcd} \Delta_g\colon X \arrow[r] \arrow[d, "f"] & X \times_Z X \arrow[d, "f \times f"] \\ \Delta_h\colon Y \arrow[r] & Y \times_Z Y \end{tikzcd}$$

By $A2$, $A3$, we have $f \times f \in \mathbf{A}$, and by $A2$ $(f \times f) \circ \Delta_g \in \mathbf{A}$. Thus $\Delta_h \circ f \in \mathbf{A}$. Now, by $A1 \Delta_{\Delta_g} \in \mathbf{A}$, thus from $A5$ it follows that $\Delta_h \in \mathbf{A}$. Then, again by $A5$, $h \in \mathbf{A}$. $\square$

In [7, Proposition 1.10] it is shown how the statement in Proposition 2.5 implies axiom $A5$.

2.6. Proposition. *Let $f = i \circ e$ be an epi-mono factorization. Then, f is in $\mathbf{A}$ if and only if i, e are in $\mathbf{A}$.*

Proof. Assume f is in $\mathbf{A}$. $A3$ implies directly that $e \in \mathbf{A}$. By $A1$ $\Delta_i \in \mathbf{A}$. Thus from $A5$ it follows $i \in \mathbf{A}$. For the other direction use $A2$. $\square$

2.7. Proposition. *For any object Z, $\mathbf{Et}/_Z \subset \mathbf{E}/_Z$ is closed under all $\mathbf{S}$-indexed colimits.*

Proof. By $A6$, it only remains to show that it is closed for coequalizers. This is a consequence of the following chain of statements:

2.7.1. The union of any $\mathbf{S}$-indexed family of etal subobjects is an etal subobject.

Proof. It follows from $A6$ and Proposition 2.6. $\square$

2.7.2. Given any equivalence relation $(\pi_1, \pi_2)\colon R \to X \times X$, if π_1, π_2 are in $\mathbf{A}$, then so is its quotient $q\colon X \twoheadrightarrow Q$.

Proof. Use that R is the kernel-pair of q and apply $A4$. $\square$

2.7.3. Let $h \in \mathbf{A}$, $h\colon Y \to Z$. Consider two arrows $f, g \in \mathbf{A}$, $f, g\colon X \to Y$, $hf = hg$, and let $(\pi_1, \pi_2)\colon R \to Y \times Y$ be the equivalence relation generated by $(f, g)\colon X \to Y \times_Z Y$.
Then, $\pi_1, \pi_2 \in \mathbf{A}$.

Proof. $A3$ implies that the projections $Y \times_Z Y \to Y$ are in $\mathbf{A}$. So by $A2$ it will be enough to show that $R \to Y \times_Z Y$ is in $\mathbf{A}$. The map $(f, g)\colon X \to Y \times_Z Y$ is in $\mathbf{A}$ by $A3$, thus so is its image $\mathrm{Im}(f, g)$ by Proposition 2.6. In the same way $\mathrm{Im}(f, g)^{\mathrm{op}} = \mathrm{Im}(g, f)$ is in $\mathbf{A}$. By $A0$ the diagonal $\Delta_h\colon Y \to Y \times_Z Y$ is in $\mathbf{A}$, thus by (a finite instance of) 2.7.1 it follows that the symmetric reflexive relation H generated by $\mathrm{Im}(f, g)$ (inside $Y \times_Z Y$) is in $\mathbf{A}$. From $A2$, $A3$, and Proposition 2.6 it follows that the composite $H \circ H$ is also in $\mathbf{A}$. 2.7.1 finishes the proof. $\square$

2.7.4. Given any coequalizer diagram

$$\begin{tikzcd} f,g\colon X \arrow[r, shift left] \arrow[r, shift right] \arrow[dr, "s"'] & Y \arrow[r, two heads, "q"] \arrow[d, "t"] & Q \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$

If $s, t \in \mathbf{A}$, then $h \in \mathbf{A}$.

Proof. From proposition 2.2(a) and $A0$ it follows that $f, g \in \mathbf{A}$, thus by 2.7.2 and 2.7.3 we have that $q \in \mathbf{A}$. Then Proposition 2.5 finishes the proof. □

Etal classes can be transported along morphisms of topoi in the following sense.

2.8. Definition–Proposition. Let $p : \mathbf{F} \to \mathbf{E}$, $p = (p^*, p_*)$, be a morphism of topoi, then

(i) Given an etal class $\mathbf{B} \subset \mathbf{F}$, the following defines an etal class $\mathbf{A} \subset \mathbf{E}$

$$h \in \mathbf{A} \Leftrightarrow p^*(h) \in \mathbf{B}. \qquad \text{We denote } \mathbf{A} = p(\mathbf{B}).$$

(ii) Given an etal class $\mathbf{A} \subset \mathbf{E}$, we denote $\mathbf{B} = p^{-1}(\mathbf{A})$ the etal class generated in $\mathbf{F}$ by the arrows of the form $p^*(h)$ with $h \in \mathbf{A}$. That is, $p^{-1}(\mathbf{A}) = \mathrm{cl}\, p^*(\mathbf{A})$.

$$\text{Then:} \quad p^{-1}p(\mathbf{B}) \subset \mathbf{B}, \quad \mathbf{A} \subset pp^{-1}(\mathbf{A}), \quad \text{and} \quad p^{-1}(\mathbf{A}) \subset \mathbf{B} \Leftrightarrow \mathbf{A} \subset p(\mathbf{B}).$$

Proof. It is very easy. □

2.9. Definition–Proposition. Given a topos $\mathbf{E}$ equipped with an etal class $\mathbf{A}$, for any object $Z \in \mathbf{E}$, the class $\mathbf{A}/_Z$ is an etal class in the topos $\mathbf{E}/_Z$ (notice that $\mathbf{A}/_Z$ is not $\mathbf{Et}/_Z$).

Let $\pi : \mathbf{E}/_Z \to \mathbf{E}$ be the canonical morphism. We have

(i) $\mathbf{A}/_Z = \pi^{-1}(\mathbf{A})$. (ii) $\mathbf{A} \subset \pi(\mathbf{A}/_Z)$.

The same holds if $\underline{A}$ is any class of arrows in $\mathbf{E}$ that satisfies the first group of axioms, (where now by $\pi^{-1}(\underline{A})$ we mean the class generated with (only) the first group of axioms by the arrows of the form $\pi^*(h)$ with $h \in \underline{A}$). We have

(iii) $\mathrm{cl}(\underline{A})/_Z = \mathrm{cl}(\underline{A}/_Z)$ (where "cl" is defined in Definition–Proposition 1.3).

Proof. (i) follows from $A3$. Given any f, g, h in $\mathbf{E}/_Z$, consider the diagram

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[r, "{(id,f)}"] \arrow[d, "h"] & X \times Z \arrow[r, "\pi"] \arrow[d, "h \times Z"] & X \arrow[d, "h"] \\ Z & Y \arrow[l, "g"] \arrow[r, "{(id,g)}"] & X \times Z \arrow[r, "\pi"] & Y \end{tikzcd}$$

Then use the fact that the squares are pull-backs in $\mathbf{E}/_Z$ and $\mathbf{E}$, respectively.

(ii) follows from (i) and Proposition 2.7.

(iii) follows from (i) and the general fact $\mathrm{cl}\, p^{-1}(\underline{A}) = p^{-1}\mathrm{cl}(\underline{A})$ which holds by formal considerations. □

3. Etal classes determined by maps between points

3.1. Definition. A class of etal maps $\mathbf{A}$ in a topos $\mathbf{E}$ is said to be *semantically determined* if there is a topos $\mathbf{F}$ and a morphism $s : \mathbf{F}^{2op} \to \mathbf{E}$ such that $\mathbf{A} = s(\mathbf{P})$, where $\mathbf{P}$ is the etal class of pull-back squares (cf 1.4).

Notice that a morphism $s : \mathbf{F}^{2op} \to \mathbf{E}$ is the same thing as a pair $p, q : \mathbf{F} \to \mathbf{E}$ and a map $\theta : q \to p$ (that is, a natural transformation $\theta : p^* \to q^*$). Thus, an arrow $h : X \to Y$ is in $\mathbf{A}$ if and only if the left square below is a pull-back:

$$\begin{tikzcd} p^*(X) \arrow[r, "p^*(h)"] \arrow[d, "\theta X"] & p^*(Y) \arrow[d, "\theta Y"] \\ q^*(X) \arrow[r, "q^*(h)"] & q^*(Y) \end{tikzcd} \qquad \begin{tikzcd} q \arrow[r] \arrow[d, "\theta"] & X \arrow[d, "h"] \\ p \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

When $\mathbf{F} = \mathbf{S}$, an element in $p^*(X)$ can be viewed (by Yoneda) as a map $p \to X$. Thus, in this case we can say, equivalently, that h is in $\mathbf{A}$ if and only if given any commutative square as the one on the right above, there exists a unique diagonal (as indicated) making both triangles commutative.

Remark. It is immediate to check that to say that an object X in $\mathbf{E}$ is etale over 1 means exactly that θX is an isomorphism.

3.2 Theorem (in collaboration with A. Kock). *For any semantically determined etal class* $\mathbf{A}$ *in a topos* $\mathbf{E}$, *and for any object* Z *in* $\mathbf{E}$, *the category* $\mathbf{Et}/_Z$ *is a topos and the inclusion* $\mathbf{Et}/_Z \subset \mathbf{E}/_Z$ *is the inverse image of a morphism* $\mathbf{E}/_Z \to \mathbf{Et}/_Z$. (*This was called theorem A in* [3], *compare also with* [6, *Corollary* 2.3].)

Proof. It is enough to consider the case $Z = 1$. In fact, the etal class $\mathbf{A}/_Z$ in $\mathbf{E}/_Z$ is also semantically determined. Let $\mathbf{A}$ be determined by $p^*, q^* : \mathbf{E} \to \mathbf{F}$, $\theta : p^* \to q^*$. Consider the functors

$$\mathbf{E}/_Z \xrightarrow{p^*} \mathbf{F}/_{p^*Z} \quad \text{and} \quad \mathbf{E}/_Z \xrightarrow{q^*} \mathbf{F}/_{q^*Z} \xrightarrow{(\theta Z)^*} \mathbf{F}/_{p^*Z},$$

where p^*, q^* have the obvious definitions, and $(\theta Z)^*$ is pulling back along θZ. They are the inverse image of two morphisms. Check that there is a map $\varepsilon : p^* \to (\theta Z)^* \circ q^*$, and that given $X \to Z$, $Y \to Z$ and any arrow $h : X \to Y$ in $\mathbf{E}/_Z$, h is etal with respect to ε if and only if it is etal with respect to θ.

Now, the previous remark shows that the diagram $\mathbf{Et} \subset \mathbf{E} \rightrightarrows \mathbf{F}$ is a (pseudo) limit of categories (called the inverter, see [8]). Since colimits of topoi are known to exist and to be computed as the corresponding (pseudo) limits of the underlying categories and inverse image functors (for an excellent treatment of this see [10]), it follows that $\mathbf{Et}$ is a topos and that the inclusion is the inverse image of a morphism $\mathbf{E} \to \mathbf{Et}$ such that the diagram $\mathbf{F} \rightrightarrows \mathbf{E} \to \mathbf{Et}$ is a co-inverter of topoi. □

3.3. Example (tiny objects). Recall that an object D in an elementary (in the sense of Lawvere) topos $\mathbf{E}$ is said to be *tiny* when the exponential functor $(-)^D : \mathbf{E} \to \mathbf{E}$ has a right adjoint $(-)_D$. (also denoted $(-)^{1/D}$). This means, for Grothendieck topoi, that $(-)^D$ is the inverse image of a geometric morphism.

Lawvere pointed out that these objects merit a serious investigation. There is a notion of *discrete object* attached to them. An object X in $\mathbf{E}$ is said to be discrete if the map $X \to X^D$ (induced by $D \to 1$) is an isomorphism.

Given a tiny object D, there is a $\mathbf{E}$-point of $\mathbf{E}$ (which we denote also by D) whose inverse image functor is the $\mathbf{E}$-model $(-)^D : \mathbf{E} \to \mathbf{E}$. For gros (Grothendieck) topoi, where $\mathbf{E} \to \mathbf{S}$ is a local geometric morphism, D also determines an $\mathbf{S}$-point of $\mathbf{E}$ (here, the representable functor $[D, -] : \mathbf{E} \to \mathbf{S}$ is the inverse image). This is the case for the gros topoi which are the models of Synthetic Differential Geometry. In that context, the object D (of square zero infinitesimals) has a section $0 : 1 \to D$, and a map $h : X \to Y$ is (some-times) said to be etal if the following square is a pull-back:

$$\begin{array}{ccc} X^D & \xrightarrow{h^D} & Y^D \\ \Big\downarrow{\scriptstyle ev_0} & & \Big\downarrow{\scriptstyle ev_0} \\ X & \xrightarrow{h} & Y \end{array}$$

(where the vertical arrows are evaluation at zero)

Clearly these etal maps form the etal class semantically determined by the map of points $0 : 1 \to D$. The discrete objects associated to D are precisely the objects etal over 1. Notice that these objects are also (by definition) the objects etal over 1 for the etal class semantically determined by the map $D \to 1$. However, these two etal classes are not the same. While the two etal topoi $\mathbf{Et}$ coincide, for a general object Z, the corresponding topoi $\mathbf{Et}/_Z$ are quite different.

Lawvere proposed the fundamental question: Consider a pair of (elementary) topoi $\mathbf{E} \to \mathbf{S}$. Which axioms express the fact that $\mathbf{E}$ is gros (relative to $\mathbf{S}$)? Which axioms on $\mathbf{E}$ will provide an (elementary) way to construct the "base" topos $\mathbf{S}$ out of $\mathbf{E}$?

Related to this, in the light of the developments considered in this paper, the problem poses itself whether Theorem 3.2 has some validity in the context of elementary topoi.

4. Infinitesimal extensions

Let $\mathbf{E}$ be any topos, $h : X \to Y$ be any arrow in $\mathbf{E}$, and $\theta : q \to p$ be any map between two $\mathbf{F}$-points of $\mathbf{E}$, $p, q : F \to E$. The square

$$\begin{array}{ccc} p^*(X) & \xrightarrow{p^*(h)} & p^*(Y) \\ \Big\downarrow{\scriptstyle \theta X} & & \Big\downarrow{\scriptstyle \theta Y} \\ q^*(X) & \xrightarrow{q^*(h)} & q^*(Y) \end{array}$$

was used in Definition 3.1 to define etal arrows h determined by θ. Obviously, this can be turned around, and used to define *infinitesimal extensions* θ determined by h.

4.1. Definition. Let $\mathbf{E}$ be any topos equipped with a class $\mathbf{A}$ of etal arrows. An arrow between points $\theta : q \to p$, $p, q : \mathbf{F} \to \mathbf{E}$ is said to be an *infinitesimal extension* if the square above is a pull-back for all h in $\mathbf{A}$. That is, if the corresponding morphism $s : \mathbf{F}^{2op} \to \mathbf{E}$ is such that $\mathbf{A} \subset s(\mathbf{P})$, where $\mathbf{P}$ is the etal class of pull-back squares. (cf. Definition 3.1 and Section 1.4).

When the map $\theta : q \to p$ is considered in the dual category as a morphism of models $p^* \to q^*$, it is some times called an *admissible morphism* [1]. Also, in some examples, a *local morphism*. In [7] it is called an *anodyne extension*. Infinitesimal extension is Joyal's original terminology, and we use it in accordance with the well-established fact that the word "etal" stands for local homeomorphism ("espace étalé" in French). In this sense, the existence of the lifting in the right hand square in Definition 3.1 means precisely that θ is an "infinitesimal" extension with respect to the etal notion satisfied by h.

Clearly, for etal classes $\mathbf{A} = s(\mathbf{P})$ semantically determined, s (actually the map θ which defines s) becomes an infinitesimal extension.

4.2. Observation. A map q as above is an infinitesimal extension provided the square is a pull-back for all arrows h in any part $\underline{A} \subset \mathbf{A}$ that generates $\mathbf{A}$ via the axioms. That is, if the corresponding morphism s is such that $\underline{A} \subset s(\mathbf{P})$.

4.3. Proposition. *A map θ as above is an infinitesimal extension if and only if the corresponding morphism s is such that $\mathbf{A} \subset s(\mathbf{Q})$, where $\mathbf{Q}$ is the etal class of quasi pull-back squares.* (cf. 1.4). *Thus, infinitesimal extensions are also those maps θ for which the square above is a quasi pull-back for all h in $\mathbf{A}$. Moreover, as in the observation above, it is enough that $\underline{A} \subset s(\mathbf{Q})$ for any part $\underline{A} \subset \mathbf{A}$ (closed under $A0$) that generates $\mathbf{A}$ via the axioms.*

Proof. It is an immediate consequence of axiom $A0$ and Proposition 1.4.1. □

4.4. Proposition. *Let Z be any object in $\mathbf{E}$, and let $\theta : q \to p$, $p, q : \mathbf{F} \to \mathbf{E}/_Z$ be a map between points of $\mathbf{E}/_Z$. Consider the canonical morphism $\pi : \mathbf{E}/_Z \to \mathbf{E}$. Then, θ is an infinitesimal extension for the etal class $\mathbf{A}/_Z$ in $\mathbf{E}/_Z$ if and only if $\pi\theta$ is an infinitesimal extension for the etal class $\mathbf{A}$ in $\mathbf{E}$.*

Proof.

$$\begin{array}{ll} \text{if } \mathbf{A}/_Z \subset s(\mathbf{P}) & \text{then } \mathbf{A} \subset \pi(\mathbf{A}/_Z) \subset \pi s(\mathbf{P}). \\ \text{if } \mathbf{A} \subset \pi s(\mathbf{P}) & \text{then } \pi^{-1}\mathbf{A} \subset \pi^{-1}\pi s(\mathbf{P}) \subset s(\mathbf{P}). \end{array}$$

This argument is justified by Definition–Propositions 2.8, 2.9 and Observation 4.2.

□

5. The construction of germs

Given a topos **E** equipped with an etal class **A**, and a point $s : \mathbf{F} \to \mathbf{E}$, we shall develop the construction of a point $t : \mathbf{F} \to \mathbf{E}$, and an infinitesimal extension $\varepsilon : s \to t$, inspired in the construction of germs of continuous functions (evaluation of the germ corresponds to $\varepsilon : t^* \to s^*$). Other conspicuous examples of this construction are the localization of a ring at a prime ideal, and the Godement construction of the etal space.

The construction, in our general setting will not yield a point unless a condition relating the class **A** with the epimorphisms in **E** holds. We shall describe this condition in terms of a site of definition. It amounts essentially to the fact that the coverings should be etal and that etal maps in the site generate (via the axioms) all etal maps in the topos. Concretely:

5.1. Condition ETC (Etal Topology Condition)**·** A topos **E** equipped with an etal class **A** satisfies *condition ETC*, if there is a subcanonical site of definition $\underline{E}$, and a class of maps $\underline{A} \subset \underline{E}$ such that

(i) $\underline{E}$ has finite limits and $\underline{A}$ satisfies the first group of axioms $A0 - A3$.

(ii) All maps in the covering families (of a pretopology) are in $\underline{A}$.

(iii) $\underline{A} \subset \mathbf{A}$, and $\mathbf{A} = \mathrm{cl}(\underline{A})$. (that is, **A** is generated by $\underline{A}$, see Definition–Proposition 1.3). (It is not required that **A** restricts to $\underline{A}$).

Remark. Given any object C in the site $\underline{E}$, condition ETC holds for the topos $\mathbf{E}/_C$ equipped with the etal class $\mathbf{A}/_C$.

Proof. (i) and (ii) are easy and follow from well-known facts. (iii) follows from Definition–Proposition 2.9(iii). $\square$

Although we shall not need it in this paper (except for the comment in 5.3) the following description is worth mentioning:

5.1.1. The etal class A generated by $\underline{A}$ (under condition ETC)· Given an arbitrary arrow h, consider a diagram as follows in the topos **E**:

$$\begin{tikzcd}
 & Z \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "pb"] & X \arrow[d, "h"] \\
C\alpha \arrow[ur, "g_\alpha"] \arrow[r, "h_\alpha"] & C \arrow[r, "y"] & Y
\end{tikzcd}$$

Then

$h \in \mathbf{A} \;\Leftrightarrow\;$ (1) For all y (with $C \in \underline{E}$), $g \in \mathbf{A}$. In turn, $g \in \mathbf{A}$ if there exists $C_\alpha \in \underline{E}$, $h_\alpha \in \underline{A}$, and liftings g_α such that the family g_α is epimorphic.

(2) The same requirement has to be satisfied by the diagonal Δ_h.

Proof. It is immediate to see from the axioms that an arrow generated as described above is in **A**. To check that these arrows form already a class is straightforward but long. The reader is warned that both conditions (ii) and (iii) in ETC are necessary. □

5.2. Theorem (this is Theorem B of Dubuc [3]). *Let* **E** *be a topos equipped with an etal class* **A**. *Assume that condition ETC holds. Then, given any morphism* $s:\mathbf{F}\to\mathbf{E}$, *we have*

(i) *The category of infinitesimal extensions below s has a terminal object. That is: there exists a morphism* $t:\mathbf{F}\to\mathbf{E}$, *and an infinitesimal extension* $\varepsilon: s\to t$ *such that given any* $u:\mathbf{F}\to\mathbf{E}$, $\phi: s\to u$, *there exists a unique infinitesimal extension* $\eta: u\to t$ *such that* $\varepsilon=\eta\circ\phi$. *Furthermore*

(ii) *For any object* $X\in\mathbf{E}$, $t^*(X)\to 1\in s^{-1}(\mathbf{A})$ (*cf. Definition* 2.8(ii)).

The proof of this theorem follows several steps:

5.2.1. Germ construction. Let $\underline{F}$ be a site of definition for **F** (such that $s^*:\underline{E}\to\underline{F}$). Denote by $\underline{\mathrm{Et}}\subset\underline{E}$ the set of objects $H\in\underline{E}$ such that $H\to 1$ is in $\underline{A}$. The germ construction determines a functor:

$$\underline{F}^{op}\times\underline{E}\xrightarrow{g}\mathbf{S}.$$

We define it as follows: Let $Z\in\underline{F}$, $X\in\underline{E}$ be any two objects. Then $g(Z,X)$ is the set

$$g(Z,X)=\{(x,H,f)\mid H\in\underline{\mathrm{Et}},\ x:Z\to s^*H, f:H\to X\}/\equiv \qquad (0)$$

where the equivalence relation "≡" is the following:
$(x,H,f)\equiv(x',H',f')$ if there exists $W\in\underline{\mathrm{Et}}$, $y:Z\to s^*W$ such that

$$\begin{tikzcd}
 & s^*H & & H \arrow[dr, "f"] & \\
Z \arrow[ur, "x"] \arrow[r, "y"] \arrow[dr, "x'"'] & s^*W \arrow[u] \arrow[d] & & W \arrow[u] \arrow[d] & X \\
 & s^*H' & & H' \arrow[ur, "f'"'] &
\end{tikzcd}$$

commute.

Notice that from Proposition 2.3 and ETC(i) it follows that the comma category $(Z,s^*|\underline{\mathrm{Et}})^{op}$ is filtered. Also, by construction of filtered colimits in **S**, $g(Z,X)$ is the colimit of the functor:

$$(Z,s^*|\underline{\mathrm{Et}})^{op}\to\mathbf{S}\quad (Z\to s^*H)\mapsto[H,X]. \qquad (1)$$

An alternative (and interesting) way of writing $g(Z,X)$ is the following coend:

$$g(Z,X) = \int^{H \in \underline{\mathrm{Et}}} [Z, s^*H] \times [H,X]. \tag{2}$$

The set $g(Z,X)$ is also the colimit of the functor:

$$(\underline{\mathrm{Et}}, X) \to S, \quad (H \to X) \mapsto [Z, s^*H]. \tag{3}$$

(We denote with square brackets the set of arrows in the corresponding category.) All this follows by well-known formulas for Kan extensions, see for example [9]. $\square$

5.2.2. (i) For each $Z \in \underline{F}$, the functor $g(Z,-) : \underline{E} \to \mathbf{S}$ is left exact.

(ii) For each cover $X_\alpha \to X$ in $\underline{E}$, and each $Z, \xi \in g(Z,X)$, there exists $Z_\alpha \to Z$ in $\underline{F}$, and $\xi_\alpha \in g(Z_\alpha, X)$ such that $Z_\alpha \to Z$ covers and

$$\begin{tikzcd} Z_\alpha \arrow[r] \arrow[d, "\xi_\alpha"] & Z \arrow[d, "\xi"] \\ g(-,X_\alpha) \arrow[r] & g(-,X). \end{tikzcd}$$

commutes.

Proof. (i) This is clear, since formula (1) in 5.2.1 says that $g(Z,-)$ is the filtered colimit of the representable functors $[H,-]$.

(ii) We shall give a sketch of the proof. Let $(x,f) = (Z \xrightarrow{x} s^*H,\ H \xrightarrow{f} X)$ be a representative of ξ. Define (x_α, f_α) by the following pull-backs:

$$\begin{tikzcd} Z_\alpha \arrow[r, "x_\alpha"] \arrow[d] & s^*H_\alpha \arrow[d] & H_\alpha \arrow[r, "f_\alpha"] \arrow[d] & X_\alpha \arrow[d] \\ Z \arrow[r, "x"] & s^*H & H \arrow[r, "f"] & X. \end{tikzcd}$$

By ETC (ii) we can assume that $X_\alpha \to X$ is in $\underline{A}$, then by Condition 5.1(i) it follows that $H_\alpha \in \underline{\mathrm{Et}}$. Then the pairs (x_α, f_α) determine the elements $\xi_\alpha \in g(Z_\alpha, X)$ in the statement. It is clear that $H_\alpha \to H$ covers, thus $s^*H_\alpha \to s^*H$ also covers, which in turn shows that $Z_\alpha \to Z$ is a covering. $\square$

5.2.3. There is a natural transformation $\varepsilon : t \to s$ (into the functor $s, s(Z,X) = [Z, s^*X]$) such that for all etal maps $h : X \to Y$ in $\underline{A}$, and all Z in $\underline{F}$, the square

$$\begin{tikzcd} g(Z,X) \arrow[r, "g(Z,h)"] \arrow[d, "\varepsilon"] & g(Z,Y) \arrow[d, "\varepsilon"] \\ {[Z,s^*X]} \arrow[r, "{[Z,s^*h]}"] & {[Z,s^*Y]} \end{tikzcd} \tag{4}$$

is a pull-back.

Proof. As before, we give a sketch of the proof. The natural transformation ε is given by composition. That is, given $\xi \in g(Z,Y)$, take a representative (y, f) and define

$$\varepsilon : g(Z,Y) \to [Z, s^*Y]:$$

$$\varepsilon(Z \xrightarrow{y} s^*H, H \xrightarrow{f} Y) = Z \xrightarrow{y} s^*H \xrightarrow{s^*f} s^*Y.$$

Now, let $z \in [Z, s^*X]$ be such that the square:

$$\begin{tikzcd} Z \arrow[r, "y"] \arrow[d, "z"] & s^*H \arrow[d, "s^*f"] \\ s^*X \arrow[r, "s^*h"] & s^*Y \end{tikzcd}$$

commutes. Define W, g as a pull-back

$$\begin{tikzcd} W \arrow[r] \arrow[d, "g"] & H \arrow[d, "f"] \\ X \arrow[r, "h"] & Y . \end{tikzcd}$$

There exists a $Z \xrightarrow{x} s^*W$ such that

$$\begin{tikzcd} & s^*H \arrow[dr, "s^*f"] & \\ Z \arrow[ur, "y"] \arrow[r, "x"] \arrow[dr, "z"'] & s^*W \arrow[u] \arrow[d] & s^*Y \\ & s^*X \arrow[ur, "s^*h"'] & \end{tikzcd}$$

commutes. It follows from ETC(i) that W is in $\underline{\mathrm{Et}}$. Therefore the pair (x, g) determines an element in $g(Z,X)$, which is used to show that the square (4) is a pull-back. $\square$

5.2.4 (corollary of 5.2.2). ***Construction of the functor*** $t^*:\mathbf{E}\to\mathbf{F}$. *Consider the composite functor* $t=\#\circ g:\underline{E}\xrightarrow{g}\mathbf{S}^{\underline{F}^{op}}\xrightarrow{\#}\mathbf{F}$, *where g corresponds to the functor (germ construction) of* 5.2.1, *and # is the associated sheaf functor. It is clear by* (i) *in* 5.2.2 *that t is left exact. On the other hand, it readily follows from* (ii) *in* 5.2.2 *that t sends coverings in $\underline{E}$ into epimorphic families in* $\mathbf{F}$ (*to prove this it may help to recall that for any presheaf F, the family $Z\xrightarrow{\xi}\#F$ (all $Z\xrightarrow{\xi}F$) is epimorphic in* $\mathbf{F}$). *It is well known (see SGA*4 [11]), *that in this case the Kan extension along the Yoneda functor (here an embedding), extends t to the topos E, and that this extension, that we denote* $t^*:\mathbf{E}\to\mathbf{F}$, *is the inverse image of a geometric morphism. Also, it is the case that t^* is the composite* $\mathbf{E}\xrightarrow{g^\sim}\mathbf{S}^{\underline{F}^{op}}\xrightarrow{\#}\mathbf{F}$, *where $g^\sim$ (which does not preserve colimits) is the Kan extension of* $\underline{E}\xrightarrow{g}\mathbf{S}^{\underline{F}^{op}}$. □

5.2.5 (observation). By definition, the functor $\underline{E}\xrightarrow{g}\mathbf{S}^{\underline{F}^{op}}$ is given by a formula $[Z,g(X)]=g(Z,X)$, for $Z\in\underline{F}, X\in\underline{E}$. Let us indicate (also) by $\mathbf{E}\xrightarrow{g}\mathbf{S}^{\underline{F}^{op}}$ the functor defined in the same way, but now with X in $\mathbf{E}$. That is, a section $Z\xrightarrow{\xi}g(X)$ is a "germ" (has representatives ($Z\xrightarrow{x}s^*H, H\xrightarrow{f}X$), for some H in $\underline{\mathrm{Et}}$, etc., see (0) in 5.2.1). Then, this construction is actually the Kan extension of itself when restricted to $\underline{E}$. That is, $g^\sim=g$. Thus, the formula $[Z,g^\sim(X)]=g(Z,X)$ holds for any X in $\mathbf{E}$. □

5.2.6. *Given any $X\in\mathbf{E}$, there is a colimit diagram in* $\mathbf{F}$:

$$\lambda_f: s^*H\to t^*X \quad (\textit{where } f \textit{ ranges over all } (H\to X)\in(\underline{\mathrm{Et}},X)).$$

Proof. By the observation above, the construction 5.2.4 and formula (3) in 5.2.1. □

5.2.7 (corollary of 5.2.3)**.** Construction of the natural transformation ε. *There is a natural transformation $\varepsilon:t^*\to s^*$ such that for all maps $h:X\to Y$ in* $\mathbf{A}$, *the square*

$$\begin{tikzcd} t^*(X) \arrow[r, "t^*(h)"] \arrow[d, "\varepsilon X"] & t^*(Y) \arrow[d, "\varepsilon Y"] \\ s^*(X) \arrow[r, "s^*(h)"] & s^*(Y) \end{tikzcd}$$

is a pull-back.

Proof. From 5.2.3 it follows that there is a natural transformation $\varepsilon: g^\sim\to i^*s^*$ (where i^* is the inclusion $\mathbf{F}\to\mathbf{S}^{\underline{F}^{op}}$) such that for all maps $h:X\to Y$ in $\underline{A}$, the corresponding square is a pull-back. By taking the associated sheaf #, we see that there is a natural transformation $\varepsilon:t^*\to s^*$ which makes the square above a pull-back for all h in $\underline{A}$. Now, since ETC (iii) says that $\underline{A}$ generates $\mathbf{A}$ by the axioms, it follows (recall Observation 4.2) that the square above is a pull-back for all $h:X\to Y$ in $\mathbf{A}$. □

5.2.8. Conclusion of the proof of Theorem 5.2. It is clear from 5.2.4. and 5.2.7. that to finish the proof of part (i) only the universal property of ε remains to be proved. This follows immediately since (in the notation in Theorem 5.2) for any H etal over 1, $\phi H : u^*(H) \to s^*(H)$ is an isomorphism. The germ construction associated either to u or to s is the same. Part (ii) follows from 5.2.6 (recall also Proposition 2.7 and Definition–Proposition 2.8(ii)). □

5.3. Comment. Given any $Z \in \underline{F}$, $X \in \mathbf{E}$, a section $\xi : Z \to t^*X$ is just a section of the sheaf associated to the presheaf $g^{\sim}X$ whose sections are germs defined on some H in $\underline{E}$ (cf. Observation 5.2.5). It is possible to see (by inspection of the classical construction of the associated sheaf and the construction (5.1.1) of the smallest etal class containing $\underline{A}$) that ξ is also a germ, but this time defined on some H in **Et**, and that any such germ is so determined. In fact, the following holds:

$$[Z, t^*X] = \int^{H \in \mathbf{Et}} [Z, s^*H] \times [H, X].$$

(in particular, the (large) coend on the right is a (small) set that lives in **S**).

6. Some immediate consequences of Theorem 5.2

The particular case of Theorem 5.2 where the point s is the identity $\mathbf{E} \to \mathbf{E}$ furnishes a strong form of the conclusion of Theorem 3.2, but only for slicing over objects in the site (see remark in Condition 5.1). Namely:

6.1. Theorem. *Let* **E** *be any topos equipped with an etal class* **A** *such that condition ETC holds. Then* **Et** *is a topos and the inclusion* $\mathbf{Et} \subset \mathbf{E}$ *is the inverse image of a local geometric morphism* $r : \mathbf{E} \to \mathbf{Et}$.

Proof. Let $t : \mathbf{E} \to \mathbf{E}$, $\varepsilon : id \to t$ be the morphism corresponding to $s = id$ in Theorem 5.2. Its inverse image t^* takes values in **Et**. Call $r^* : \mathbf{E} \to \mathbf{Et}$ the functor defined by $r_*(X) = t^*(X)$. Let r^* be the inclusion $\mathbf{Et} \to \mathbf{E}$. Thus $t^* = r^* r_*$, and we have $\varepsilon : r^* r_* \to id$. Since ε is infinitesimal it follows that $\varepsilon r^* : r^* r_* r^* \to r^*$ is an isomorphism. This shows that r_* is right adjoint to r^*. Clearly r^* preserves all colimits. Thus, by 1.4 in [6] the pair r^*, r_* defines a local geometric morphism $r : \mathbf{E} \to \mathbf{Et}$. Finally, from the preservation of colimits by r^* is easy to check that the images of the generators for **E** are generators for **Et**. □

Let $i : \mathbf{Et} \to \mathbf{E}$ be the *center* (in the sense of [6]) of the local geometric morphism in Theorem 6.1. That is, $i_*(H) = r^*(H) = H$, and $i^*(X) = r_*(X) = t^*(X)$. We have

6.2. Proposition. *Let* **E** *be any topos equipped with an etal class* **A** *such that condition ETC hold, and let* $s : \mathbf{F} \to \mathbf{E}$ *be any morphism. Let* $t : \mathbf{F} \to \mathbf{E}$ *be the morphism corresponding to s in Theorem* 5.2. *Then* $t : \mathbf{F} \to \mathbf{E}$ *factors through* **Et**, $t = i \circ t : \mathbf{F} \to \mathbf{Et} \to \mathbf{E}$.

Proof. It follows from 5.2.6 since s^* preserves colimits. □

In the situation of Theorem 6.1, it is clear from 5.2.6 that the objects in the site $\underline{E}$ which are etal over 1 generate the etal topos **Et**. It follows that $\underline{\mathrm{Et}}$ is a site of definition for **Et**. Thus, Theorem 6.1 shows that the inclusion $\underline{\mathrm{Et}} \subset \underline{E}$ induces a local geometric morphism between the topoi of sheaves $\mathrm{Sh}(\underline{E})$ and $\mathrm{Sh}(\underline{\mathrm{Et}})$. This result is (essentially) established in the "Médaille en Chocolat" exercise, SGA 4 IV 4.10.6. [11]. See also [2, 3, 7].

When the class $\underline{A} \subset \underline{E}$ in condition ETC has all of its maps being monomorphisms, it clearly follows from the equality $\mathbf{Et} = \mathrm{Sh}(\underline{\mathrm{Et}})$ that **Et** is a *spatial* topos, namely, the topos of sheaves on the locale of etal subobjects of 1. Thus:

6.3. Proposition. *Let* **E** *be any topos equipped with an etal class* **A** *such that condition ETC holds. Then* $\mathbf{Et} = \mathrm{Sh}(\underline{\mathrm{Et}})$, *and if all the maps in* $\underline{A}$ *are monomorphisms,* **Et** *is a spatial topos.* □

7. Completeness of the axioms

7.1. Definition–Proposition. Given a topos **E** and a family of maps $\underline{A} \subset \mathbf{E}$, let **A** be the etal class generated by $\underline{A}$. Given any topos **F**, an arrow h in **E** is said to be **F**-*semantically etal* if $h \in s(\mathbf{P})$ for every infinitesimal extension $s : \mathbf{F}^2 \to \mathbf{E}$, and *semantically etal* when it is **F**-semantically etal for all **F**. The class $\mathbf{A}^\sim$ of semantically etal arrows is an etal class, thus $\mathbf{A} \subset \mathbf{A}^\sim$.

A completeness theorem says that $\mathbf{A}^\sim \subset \mathbf{A}$. This completeness means that the axioms are sufficient to generate all semantically etal arrows. Also, in a sense, that there are sufficiently many infinitesimal extensions. When Joyal introduced the axioms in the late 1970s, an important problem at the time was to prove a completeness theorem. In 1987 at the Louvain la Neuve Conference we presented a completeness theorem with a proof based on Theorem 5.2 (see [3]). Joyal had also suggested a line of proof based on a concrete construction of the classifying topos for infinitesimal extensions, and he gave an explicit description of a site for this topos (4.2.5. below). Using this, in 1990 he proved a completeness theorem (in collaboration with Moerdijk) for classes which satisfy an additional axiom (see [7]). Condition ETC implies this axiom, so that our result follows. However our proof for etal classes satisfying ETC (Theorem 7.3 below) is completely different, and is based on the supply of infinitesimal extensions provided by Theorem 5.2. This theorem implies that there are enough infinitesimal extensions.

7.2. Proposition. *Let* **E** *be a topos and* **A** *an etal class in* **E**. *Given any commutative triangle*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

If h is **F**-*semantically etal with respect to* **A** *in* **E**, *then it is also* **F**-*semantically etal with respect to* $\mathbf{A}/_Z$ *in* $\mathbf{E}/_Z$. *Thus*, $h \in \mathbf{A}^{\sim} \Rightarrow h \in (\mathbf{A}/_Z)^{\sim}$.

Proof. Let $s : \mathbf{F}^{\mathbf{2}} \to \mathbf{E}/_Z$ be any infinitesimal extension, and let $\pi : \mathbf{E}/_Z \to \mathbf{E}$ be the canonical morphism. By 4.4 $\pi \circ s$ is infinitesimal, and so $h \in \pi \circ s(\mathbf{P})$. Then $\pi^* \circ h \in s(\mathbf{P})$. Considering now the left pull-back square in Definition–Proposition 2.9, it follows that $h \in s(\mathbf{P})$. □

7.3. Theorem (completeness)**.** *Let* **E** *be a topos*, *and* **A** *an etal class in* **E** *such that condition ETC holds. Then*, *all semantically etal arrows are in* **A**. *That is*, $\mathbf{A}^{\sim} \subset \mathbf{A}$.

Proof. Let $h : X \to Z$ be semantically etal. Consider all maps $C \to Z$ with C in the site $\underline{E}$. Let the following be pull-back diagrams:

$$\begin{tikzcd} D \arrow[r] \arrow[d, "f"] & X \arrow[d, "h"] \\ C \arrow[r] & Z \end{tikzcd}$$

Since the family $C \to Z$ is epimorphic, it clearly follows from axioms $A4$ and $A6$ that to show that h is in **A** it is enough to show that f is in **A** for all f. Since $h \in \mathbf{A}^{\sim}$, also $f \in \mathbf{A}^{\sim}$. By Proposition 7.2 (the particular case when $Y = Z$) it follows that f is semantically etal over 1 with respect to $\mathbf{A}/C$ as an object in $\mathbf{E}/C$. The etal class $\mathbf{A}/C$ in the topos $\mathbf{E}/C$ also satisfies ETC (see remark in 5.1). Let ε be the infinitesimal extension constructed in Theorem 5.2 associated to the identity $\mathbf{E}/C \to \mathbf{E}/C$. Then ε(f) is an isomorphism. This shows (by 5.2 (ii)) that f is etal (over 1) for the class $\mathbf{A}/C$ as an object in $\mathbf{E}/C$. Clearly this means that $f \in \mathbf{A}$. □

When the topos **E** has enough points (defined in **S**), then it suffices to consider only these points to have a completeness theorem.

7.4. Theorem (completeness with respect to **S**)**.** *Let* **E** *be a topos with enough points*, *and* **A** *an etal class in* **E** *such that condition ETC holds. Given any arrow* $h : X \to Z$, *if* $h \in s(\mathbf{P})$ *for every infinitesimal extension* $s : \mathbf{S}^{\mathbf{2}} \to \mathbf{E}$, *then* $h \in \mathbf{A}$.

Proof. The proof follows exactly the same steps as in Theorem 7.3. It is necessary now to show that $\varepsilon(f)$ is an isomorphism. Consider a point $p : \mathbf{S} \to \mathbf{E}/C$, then by Proposition 7.2 $p^*(\varepsilon(f))$ is an isomorphism. Since $\mathbf{E}/C$ also has enough points (SGA4 [11]), the result follows. □

8. The infinitesimal extension classifier

Recall that the topos of sheaves over the Sierpinski space coincides with the presheaf topos $\mathbf{S}^{\mathbf{2}^{op}}$, and that for any topos $\mathbf{F}, \mathbf{F}^{\mathbf{2}^{op}} = \mathbf{F} \times \mathbf{S}^{\mathbf{2}^{op}}$. We shall denote $\mathbf{M} = \mathbf{S}^{\mathbf{2}^{op}}$. Thus

$\mathbf{F}^{2^{op}} = \mathbf{F} \times \mathbf{M}$. It follows that a morphism $\mathbf{F} \times \mathbf{M} \to \mathbf{E}$ is a pair of $\mathbf{F}$-points of $\mathbf{E}$ with a map in between. But this corresponds to a morphism $\mathbf{F} \to \mathbf{E}^{\mathbf{M}}$. Thus, $\mathbf{E}^{\mathbf{M}}$ classifies maps between points (notice that since $\mathbf{M}$ is a presheaf topos, it is exponentiable [5]).

8.1. Definition of notation. The data corresponding to the 'evaluation' morphism $\mathbf{E}^{\mathbf{M}} \times \mathbf{M} \to \mathbf{E}$ is the *generic map* between points of $\mathbf{E}$:

$$\mu : \partial_0 \to \partial_1, \qquad \partial_0, \partial_1 : \mathbf{E}^{\mathbf{M}} \to \mathbf{E}.$$

Given any map between points $\theta : q \to p$, $q, p : \mathbf{F} \to \mathbf{E}$, there exists $t : \mathbf{F} \to \mathbf{E}^{\mathbf{M}}$ such that $\mu t = \theta$ (in the appropriate universal way).

Any subtopos $\mathbf{G}$ of $\mathbf{E}^{\mathbf{M}}$ determines a class of maps between points of $\mathbf{E}$, namely, those θ as above for which the corresponding t factors through $\mathbf{G}$. The following is the construction of the subtopos that classifies infinitesimal extensions:

8.2. Proposition (Joyal, see [3,7]). *Let* $\mathbf{E}$ *be a topos equipped with an etal class* $\mathbf{A}$. *Each arrow* $h : X \to Y$ *in* $\mathbf{E}$ *determines the following diagram in the morphism classifier* $\mathbf{E}^{\mathbf{M}}$:

$$\begin{tikzcd}
\partial_1^*(X) \arrow[rr, "\partial_1^*(h)"] \arrow[dr, "\sigma h"] \arrow[dd, "\mu X"'] & & \partial_1^*(Y) \arrow[dd, "\mu Y"] \\
& P \arrow[ur] & \\
\partial_0^*(X) \arrow[rr, "\partial_0^*(h)"] \arrow[ur] & & \partial_0^*(Y)
\end{tikzcd}$$

where the lower triangular square is a pull back. Take the topology that forces the invertibility of the arrows σ_h *for all* h *in* $\mathbf{A}$. *Then, the subtopos of sheaves, that we denote* $\mathbf{E}^{(\mathbf{M})} \subset \mathbf{E}^{\mathbf{M}}$, *classifies infinitesimal extensions. The composite of the inclusion with* $\mu : \partial_0 \to \partial_1$ *is the generic infinitesimal extension, which by abuse of notation we shall denote with the same letters*:

$$\mu : \partial_0 \to \partial_1, \qquad \partial_0, \partial_1 : \mathbf{E}^{(\mathbf{M})} \to \mathbf{E}.$$

Proof. It is clear by definition that μ becomes infinitesimal in $\mathbf{E}^{(\mathbf{M})}$. On the other hand, given an infinitesimal extension $\theta : q \to p$ (as in Definition 8.1), the inverse image t^* of the corresponding t inverts all the σ_h. Thus, it factors through the associated sheaf functor. □

8.3. Morphisms between enriched points. Given a morphism of topoi, $\mathbf{Z} \xrightarrow{t} \mathbf{E}$, we say that a point of $\mathbf{E}, \mathbf{F} \xrightarrow{p} \mathbf{E}$ is *enriched over* $\mathbf{Z}$ or $\mathbf{Z}$-*enriched* if there is a factorization $\mathbf{F} \xrightarrow{q} \mathbf{Z}, t \circ q = p$.

We can consider morphisms and infinitesimal extensions between enriched points and the corresponding classifying topoi.

8.3.1. Definition. Consider a topos $\mathbf{E}$ equipped with an etal class $\mathbf{A}$, and two (fixed) morphisms of topoi $\mathbf{Z} \xrightarrow{w} \mathbf{E} \xleftarrow{v} \mathbf{G}$. An infinitesimal extension θ from a $\mathbf{Z}$-enriched point into a $\mathbf{G}$-enriched point consists of a square as follows:

$$\begin{tikzcd} \mathbf{F} \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\theta\nearrow"] & \mathbf{G} \arrow[d, "v"] \\ \mathbf{Z} \arrow[r, "w"'] & \mathbf{E} \end{tikzcd} \qquad \theta\colon w\circ q \longrightarrow v\circ p$$

The topos $[\mathbf{Z},\ \mathbf{G}]^{(\mathbf{M})}$ is defined to be the topos that classifies these extensions. It comes equipped with a pair of morphisms ∂_0, ∂_1, and an infinitesimal extension μ as indicated below:

$$\begin{tikzcd} {[\mathbf{Z},\ \mathbf{G}](\mathbf{M})} \arrow[r, "\partial_1"] \arrow[d, "\partial_0"'] \arrow[dr, phantom, "\mu\nearrow"] & \mathbf{G} \arrow[d, "v"] \\ \mathbf{Z} \arrow[r, "w"'] & \mathbf{E} \end{tikzcd} \qquad \mu\colon w\circ\partial_0 \longrightarrow v\circ\partial_1$$

By definition, given any square as in (1), there exists a morphism $s:\mathbf{F}\to[\mathbf{Z},\mathbf{G}]^{(\mathbf{M})}$ such that $\mu s=\theta$ (in the appropriate universal way). Thus, for example, $\mathbf{E}^{(\mathbf{M})}=[\mathbf{E},\mathbf{E}]^{(\mathbf{M})}$.

The relation of this construction to the spectrum problem (see Definition 9.1) becomes very clear if we actually describe its 2-categorical aspects:

Together with s, there exist also natural isomorphisms $\alpha_0\colon q\approx\partial_0\circ s$, $\alpha_1\colon\ p\approx\partial_1\circ s$, such that $\mu s\circ w\alpha_0=v\alpha_1\circ\theta$ in the appropriate universal way (instead of $\mu s=\theta$).

The natural general context for the proposition below is that of an iteration of enrichments, but we choose to state only the particular case we need.

8.3.2. Proposition. *The infinitesimal-extension-between-enriched-points classifier always exists and can be constructed out of the infinitesimal extension classifier* $\mathbf{E}^{(\mathbf{M})}$ *by means of pull-back of topoi as indicated in the following diagram* (*the labels a, b, c, d, g, and h are for use in Theorem* 9.3):

$$\begin{tikzcd} {[\mathbf{Z},\ \mathbf{G}](\mathbf{M})} \arrow[r, "h"] \arrow[d, "g"'] \arrow[dr, phantom, "pb"] & \mathbf{P1} \arrow[r, "d"] \arrow[d, "c"] \arrow[dr, phantom, "pb"] & \mathbf{G} \arrow[d, "v"] \\ \mathbf{P0} \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "pb"] & \mathbf{E}^{(\mathbf{M})} \arrow[r, "\partial_1"] \arrow[d, "\partial 0"] \arrow[dr, phantom, "\mu\nearrow"] & \mathbf{E} \arrow[d, "id"] \\ \mathbf{Z} \arrow[r, "w"'] & \mathbf{E} \arrow[r, "id"'] & \mathbf{E} \end{tikzcd}$$

As usual we abuse the language and denote with the same letter (in all the classifiers considered) the domain and codomain arrows, as well as the generic cell. It is clear how these data are determined for $[\mathbf{Z},\mathbf{G}]^{(\mathbf{M})}$ *by the corresponding data for* $\mathbf{E}^{(\mathbf{M})}$.

Proof. Tiresome but straightforward. □

9. A general theory of spectrum

We shall consider now a general problem of spectrum associated with a morphism of topoi $w : \boldsymbol{Z} \to \mathbf{R}$, and etal classes $\mathbf{A}$ in $\mathbf{Z}$, $\mathbf{B}$ in $\mathbf{R}$, such that $\mathbf{B} \subset w(\mathbf{A})$.

Given any topos $\mathbf{T}$ equipped with an etal class $\mathbf{H}$, let $(\mathbf{TOP},\mathbf{T})_{\mathbf{H}}$ be the 2-category whose objects are morphisms $p : \mathbf{E} \to \mathbf{T}$. A 1-arrow between $p : \mathbf{E} \to \mathbf{T}$ and $q : \mathbf{F} \to \mathbf{T}$ is a pair (f,ϕ), where $f : \mathbf{E} \to \mathbf{F}$ is a morphism and $\phi : p \to q \circ f$ is an infinitesimal extension with respect to $\mathbf{H}$, $(\phi : f^* \circ q^* \to p^*)$. A 2-cell between (g,Ψ) and (f,ϕ) is a map $\theta : g \to f (\theta : f^* \to g^*)$ such that $\phi = q\theta \circ \Psi$:

$$\begin{tikzcd}
\mathbf{E} \arrow[rr, "f"] \arrow[dr, "p"'] & \arrow[d, phantom, "\overset{\phi}{\curvearrowright}"] & \mathbf{F} \arrow[dl, "q"] \\
& \mathbf{T} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& p \arrow[dl, "\psi"'] \arrow[dr, "\phi"] & \\
q \circ g \arrow[rr, "q^{\theta}"] & & q \circ f
\end{tikzcd}$$

9.1. Definition. A morphism of topoi $w : \mathbf{Z} \to \mathbf{R}$ equipped with etal classes $\mathbf{A} \subset \mathbf{Z}$, $\mathbf{B} \subset \mathbf{R}, \mathbf{B} \subset w(\mathbf{A})$, clearly induces a 2-functor $(\mathbf{TOP},\mathbf{Z})_{\mathbf{A}} \to (\mathbf{TOP},\mathbf{R})_{\mathbf{B}}$. The *problem of spectrum* consists in the construction of a right adjoint for this functor. (In the dual categories of models (inverse image functors and natural transformations) this is a *forgetful functor*, and what we are looking for is a left adjoint).

Given a point $v : \mathbf{E} \to \mathbf{R}$, the spectrum (of v) is (by definition) a point $d : \mathbf{Spc}(v) \to \mathbf{Z}$ equipped with a morphism c and an infinitesimal extension (with respect to $\mathbf{B}$) η as indicated below:

$$\begin{tikzcd}
\mathbf{Spc(v)} \arrow[r, "c"] \arrow[d, "d"'] & \mathbf{E} \arrow[d, "v"] \\
\mathbf{Z} \arrow[r, "w"'] \arrow[ur, phantom, "\overset{\eta}{\curvearrowright}"] & \mathbf{R}
\end{tikzcd}
\qquad \eta : w \circ d \longrightarrow v \circ c.$$

Given any other similar data

$$\begin{tikzcd}
\mathbf{F} \arrow[r, "p"] \arrow[d, "q"'] & \mathbf{E} \arrow[d, "v"] \\
\mathbf{Z} \arrow[r, "w"'] \arrow[ur, phantom, "\overset{\theta}{\curvearrowright}"] & \mathbf{R}
\end{tikzcd}
\qquad \theta : w \circ q \longrightarrow v \circ p.$$

there exists a morphism $t:\mathbf{F}\to\mathbf{Spc}(v)$, a natural isomorphism $\lambda: p\approx c\circ t$, and an infinitesimal extension (with respect to $\mathbf{A}$) $\beta: q\to d\circ t$, such that $\eta t\circ wb = v\lambda\circ\theta$ (in the appropriate universal way).

Notice that other than the differences in emphasis, this is formally the same as the Definition in 8.3.1, except that here β is not required to be an isomorphism. It is this last detail that makes a whole different story.

9.2. Theorem. (construction of the spectrum). *In the situation in Definition* 9.1, *let* $v:\mathbf{E}\to\mathbf{R}$ *be any object in* $(\mathbf{TOP},\mathbf{R})_{\mathbf{B}}$. *Consider the following square* (*cf. Definition* 8.3.1):

$$\begin{tikzcd}
\mathbf{Spc(v)} \arrow[r, "i"] & {[\mathbf{Z},\mathbf{E}]^{(\mathbf{M})}} \arrow[r, "\partial_1"] \arrow[d, "\partial_0"'] \arrow[dr, phantom, "\mu\nearrow"] & \mathbf{E} \arrow[d, "p"] \\
 & \mathbf{Z} \arrow[r, "w"] & \mathbf{R}
\end{tikzcd} \qquad \mu: w\circ\partial_0 \longrightarrow p\circ\partial_1.$$

(*here* μ *is infinitesimal with respect to the class* $\mathbf{B}$)

Let $\mathbf{H}\subset[\mathbf{Z},\mathbf{E}]^{(\mathbf{M})}$ *be the etal class* $\mathbf{H}=\partial_1^{-1}(\mathbf{E})\cup\partial_0^{-1}(\mathbf{A})$, *where* $\mathbf{E}$ *is the class for which all arrows in* $\mathbf{E}$ *are etale, and "$\cup$" denotes the supremum in the poset of etal classes* (*cf. Definition–Proposition* 2.8 *for the definition of inverse image of etal classes*).

Assume that condition ETC (*cf. Condition* 5.1) *holds for* $\mathbf{H},[\mathbf{Z},\mathbf{E}]^{(\mathbf{M})}$.

Let $i:\mathbf{Et}\to[\mathbf{Z},\mathbf{E}]^{(\mathbf{M})}$ *be the inclusion of the etal topos* (*cf. Proposition* 6.2). *Then,* $\mathbf{Spc}(v)=\mathbf{Et}$, $d=\partial_0\circ i$, $c=\partial_1\circ i$, *and* $\eta=\mu i$, *is the spectrum of* v.

Proof. Given any square as in (1) in Definition 9.1, let $s:\mathbf{F}\to[\mathbf{Z},\mathbf{E}]^{(\mathbf{M})}$ be as in Definition 8.3.1, and let $t:\mathbf{F}\to[\mathbf{Z},\mathbf{E}]^{(\mathbf{M})}$, $\varepsilon: s\to t$ be the morphism and infinitesimal extension given by Theorem 5.2. By Proposition 6.2 t factors $t:\mathbf{F}\to\mathbf{Spc}(v)$, $t=i\circ t$. Notice now that given any infinitesimal extension (with respect to the class $\mathbf{H}$) ε as above, the 2-cell $\partial_1\varepsilon$ is an isomorphism, and the 2-cell $\partial_0\varepsilon$ is an infinitesimal extension (with respect to the class $\mathbf{A}$). The rest of all the necessary data are explicitly given in Definitions 8.3.1 and 9.1, and the proof consists in a straightforward play with the pertinent universal properties. $\square$

The etal classes which appear associated with the problem of spectrum satisfy condition ETC by their very definition. However, given a morphism of topoi as in Theorem 9.2, to apply the theorem in practice it is necessary to show that the etal class $\mathbf{H}$ in the construction of the spectrum satisfies condition ETC. This is actually the case, but it needs some proof based on the explicit constructions developed in Appendix A. We have

9.3. Theorem. (existence of the spectrum). *Let* $w : \mathbf{Z} \to \mathbf{R}$ *be a morphism of topoi equipped with etal classes* $\mathbf{A} \subset \mathbf{Z}$, $\mathbf{B} \subset \mathbf{R}$ *satisfying condition ETC*, $\mathbf{B} \subset w(\mathbf{A})$. *Then the induced* 2*-functor* $(\mathbf{TOP}, \mathbf{Z})_{\mathbf{A}} \to (\mathbf{TOP}, \mathbf{R})_{\mathbf{B}}$ *has a right adjoint. Moreover, it can be constructed as in Theorem* 9.2.

Proof. It suffices to show that the class $\mathbf{H}$ in Theorem 9.2 satisfies condition ETC. We shall use the construction of $[\mathbf{Z}, \mathbf{E}]^{(\mathbf{M})}$ given in Proposition 8.3.2. In the diagram there, consider the etal class $\mathbf{L} \subset \mathbf{R}^{(\mathbf{M})}$, $\mathbf{L} = \partial_1^{-1}(\mathbf{R}) \cup \partial_0^{-1}(\mathbf{B})$. To see that this class satisfies ETC we look at the construction of $\mathbf{R}^{(\mathbf{M})}$ given in Proposition A.2.4. By Definition–Proposition 2.8 (ii), and axiom $A3$, the covers of type (1) and (2) (see Proposition A.2.3) are in $\mathbf{L}$. It remains to check that the covers of type (3) in the diagram in Proposition A.2.4 also are in $\mathbf{L}$. This is so since the arrows σ_h go between two etal objects over $\partial_1^*(Y)$ (use then Proposition 2.3). Consider now the construction of pull-backs of topoi given in Propositions A.1.1. and A.1.2. It follows that the classes $\mathbf{H}_0 = b^{-1}(\mathbf{L}) \cup a^{-1}(\mathbf{A})$ in $\mathbf{P}_0$ and $\mathbf{H}_1 = d^{-1}(\mathbf{E}) \cup c^{-1}(\mathbf{L})$ in $\mathbf{P}_1$ satisfy ETC. Then, by the same reasons the class $\mathbf{K} = h^{-1}(\mathbf{H}_1) \cup g^{-1}(\mathbf{H}_0)$ satisfies ETC. Finally, one can readily check that $\mathbf{H} = \mathbf{K}$ (recall that $w^{-1}(\mathbf{B}) \subset \mathbf{A}$). $\square$

9.4. Comment. The general notion of Spectrum now in use, the *Cole Spectrum*, has as ingredients a geometric theory, and a quotient theory together with a class of "admissible" morphisms between its models. This class is supposed to satisfy an axiom called *the factorization lemma*. The spectrum is then obtained by means of certain topos theoretical constructions [5]. A draw-back of this approach is that when it is applied to the examples, all the work has to be done again to show that the general construction produces the known original construction of the example (this is the case notably with the classical spectra constructed by Hakim in [4]). Even worse, in addition, in practice one also has to prove the validity of the factorization lemma in each case. However, Coste in [1] has developed a context (that includes all the known examples) where he can prove the factorization lemma with sufficient generality.

The theory we develop here has several good points in its favor:

(a) It is more general. Here we have any morphism of topoi, not just the inclusion of a sub-topos (as in a quotient theory).

(b) Conceptually, it frees the notion of spectrum from the notion of geometric theory (in the sense that etal classes take care of the role assigned to the admissible morphisms). Attention is focused in the topos rather than in its points.

(c) The construction involved (9.2) reflects the geometric nature of the problem, and, when applied to the examples it gives on the nose the original constructions of Hakim (cf. Example 9.5).

(d) There is no need to prove anything (like the factorization lemma) in practice.

(e) It includes all the examples (in particular the context and the theorem of Coste).

9.5. Example (*Zariski and Etal topoi*)**.** These are spectra situations arising in Algebraic Geometry. Consider a (fixed) ring K and the category C of affine schemes (dual of the category of finitely presented K-algebras). The trivial topology defines the presheaf topos $\mathbf{R}$. We consider in $\mathbf{R}$ the etal class $\mathbf{B}$ generated by the isomorphisms (smallest etal class). The Zariski open covers determine the topology in C that defines the topos $\mathbf{Z}$. We consider in $\mathbf{Z}$ the etal class $\mathbf{A}$ generated by the Zariski opens. Clearly in both cases condition ETC is satisfied. Thus Theorem 9.3 applies. Given any finitely presented K-algebra $A:\mathbf{S}\to\mathbf{R}$, its spectrum is a spatial topos by Proposition 6.3. Let C be the affine scheme defined by A. It is immediately checked that the arrow $[\mathbf{Z},\mathbf{S}]^{(\mathbf{M})} \xrightarrow{\partial_0} \mathbf{Z}$ in Theorem 9.2 (which is in this case the *generic local A-algebra*) is given by the slice topos and the canonical morphism $\mathbf{Z}/_C \xrightarrow{\pi} \mathbf{Z}$. Since now $\mathbf{E}$ is the topos $\mathbf{S}$, the arrow $[\mathbf{Z},\mathbf{S}]^{(\mathbf{M})} \xrightarrow{\partial_1} \mathbf{E}$ is the only morphism $\mathbf{Z}/_C \to \mathbf{S}$, and the arrows in the class $\partial_1^{-1}(\mathbf{E})$ are all maps between constant sheaves, which are always etal. Thus, the etal class $\mathbf{H}$ is just $\mathbf{H}=\pi^{-1}(\mathbf{A})$. It follows from Definition–Proposition 2.9 that $\mathbf{H}$ is generated by the Zariski opens in $\mathbf{Z}/_C$. Thus, the construction in Theorem 9.2, together with Proposition 6.3, show that the spectrum of A (or C) is the classically considered [4] (small) topos of sheaves for the Zariski covers of C.

Exactly in the same way, but considering the etal covers in C to define the topos $\mathbf{Z}$, and the etal morphisms to generate the class $\mathbf{A}$, we see that the spectrum of A (or C) is in this case the classical etal topos of C, namely, the topos of sheaves for the etal covers of C.

Although Theorems 9.2 and 9.3 apply directly to the examples, for the sake of completeness we include the following:

9.6. Example (Cole spectrum). As mentioned above, this is the particular case where $\mathbf{R}$ is the classifying topos of a geometric theory $\mathbf{S}, \mathbf{Z}$ is the classifying topos of a quotient theory $\mathbf{T}$, and the etal class $\mathbf{A}$ in $\mathbf{Z}$ is semantically determined by a (axiomatic, including the factorization lemma) class of *admissible* morphisms. Then, a construction based on the factorization lemma (and different from ours) produces the spectrum [5]. Coste considers a context (*factorization triples* [1]) where he shows the validity of the factorization lemma. Factorization triples clearly determine topoi equipped with etal classes which define the admissible morphisms (as the corresponding infinitesimal extensions), and which satisfy by their very definition condition ETC. The following is a proof of the factorization lemma for factorization triples:

9.6.1. Theorem (Coste). *Consider the topos in Definition* 8.3.1 (*associated to a factorization triple*):

$$\begin{tikzcd}
{[\mathbf{Z},\mathbf{R}]^{(\mathbf{M})}} \arrow[d, "\partial_0"'] \arrow[dr, "\partial_1"] & \\
\mathbf{Z} \arrow[r, "w"'] \arrow[ur, phantom, "\mu" near start] & \mathbf{R}
\end{tikzcd} \qquad \mu\colon w\circ\partial_0 \longrightarrow \partial_1.$$

(*now* μ *is the generic morphism from a* **S**-*model into a* **T**-*model, recall that by Grothendieck's convention arrows between models go the other way*).

Condition ETC is checked for the class $\mathbf{H} = \partial_1^{-1}(\mathbf{R}) \cup \partial_0^{-1}(\mathbf{A})$ in the same way as in Theorem 9.3. Then apply Theorem 5.2 to the identity morphism $\varepsilon : id \to t : [\mathbf{Z}, \mathbf{R}]^{(\mathbf{M})} \to [\mathbf{Z}, \mathbf{R}]^{(\mathbf{M})}$. As before, $\partial_1 \varepsilon$ is an isomorphism and $\partial_0 \varepsilon$ is an infinitesimal extension (with respect to the class **A**). This gives exactly the factorization lemma for the generic morphism.

Appendix A. (some constructions with topoi in terms of sites)

A.1. Pull-backs of topoi

A.1.1. Proposition. *Consider a push-out diagram of categories with finite limits and left exact functors*:

$$\begin{array}{ccc} \underline{H} & \xrightarrow{p^*} & \underline{F} \\ \uparrow{\scriptstyle t^*} & & \uparrow{\scriptstyle q^*} \\ \underline{E} & \xrightarrow{s^*} & \underline{G} \end{array}$$

Assume there are topologies in $\underline{E}, \underline{H}, \underline{G}$, and $\underline{F}$ such that the arrows determine morphisms of sites, and that the topology in F is the coarsest which makes p^* and q^* continuous (this always makes p^* and q^* morphisms of sites). Then, the corresponding square between the categories of sheaves is a pull-back of topoi.

Proof. In order to test if the square is a pull-back, consider a topos **X** and a pair of morphisms $\mathbf{X} \xrightarrow{h} \mathbf{H}$, $\mathbf{X} \xrightarrow{g} \mathbf{G}$. Let $f^* : F \to \mathbf{X}$, $f^* \circ p^* = h^*$, $f^* \circ q^* = g^*$ be the left exact functor determined by the universal property of push-outs. Consider the topology in $\underline{F}$ whose covering families are those families which are sent by f^* into epimorphic families in **X**. Clearly, by the assumptions made, this topology contains the topology of the site $\underline{F}$. Thus f^* is continuous. Then, an immediate application of [11, SGA4 IV 4.9.4] finishes the proof. □

Remark. If we define full subcategories $\underline{F}_0 = p^*(\underline{H}) \cup q^*(\underline{G})$, $\underline{F}_n \subset \underline{F}_{n+1}$, where in $\underline{F}_{n+1}$ we put all pull-backs of diagrams in $\underline{F}_n$, then $\underline{F}$ is the (denumerable) union of all the $\underline{F}_n$ (notice that $1 \in \underline{F}_0$). However we will not have any use of this fact here.

Corollary (existence of pull-back of topoi). *We shall call the topology in* $\underline{F}$, (*the coarsest which makes* p^* *and* q^* *continuous*) *the product topology. Any pair of* *morphisms between topoi* $\mathbf{H} \to \mathbf{E} \leftarrow \mathbf{G}$ *can be presented by a pair* $\underline{H} \leftarrow \underline{E} \to \underline{G}$ *of left exact functors between (small) sites with finite limits. In view of A.1.4 below, we can construct the push-out of categories with finite limits and limit preserving functors* $\underline{H} \to \underline{F} \leftarrow \underline{G}$, *and furnish* $\underline{F}$ *with the product topology. In this way, by Proposition A.1.1, we obtain a site for the pull-back topos, showing at the same time its existence.*

A.1.2. Proposition. *In the situation of Proposition* A.1.1, *consider generating pretopologies in* $\underline{H}$ *and* $\underline{G}$. *Then, the class of families* $X_\alpha \to X$ *in* $\underline{F}$ *obtained by pulling-back pretopology covers as indicated below*:

$$\begin{array}{ccccccccc} X & \longrightarrow & p^*(Y) & & Y & \qquad & X & \longrightarrow & q^*(Z) & & Z \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ X_\alpha & \longrightarrow & p^*(Y_\alpha) & & Y_\alpha & & X_\alpha & \longrightarrow & q^*(Z_\alpha) & & Z_\alpha \end{array}$$

is stable by pull-backs, and if we close it by composition we obtain a pretopology that generates the product topology.

Proof. It is straightforward. □

A.1.3. Example (*product of topoi*). When $\underline{E} = 1$, so that $\mathbf{E} = \mathbf{S}$ is the terminal topos, we have the product topos $\mathbf{F} = \mathbf{H} \times \mathbf{G}$. In this case a push-out $\underline{H} \to \underline{F} \leftarrow \underline{G}$ is given by the product of categories $\underline{F} = \underline{H} \times \underline{G}$, and every object X in $\underline{F}$ can be identified with a product $X = Y \times Z$, with Y in $\underline{H}$ and Z in $\underline{G}$. The covers of X are refined by products of covers of Y and of Z. If $\mathbf{H}$ and $\mathbf{G}$ are spatial topoi, then we get the familiar base for the product topology (here $\underline{F}$ is a set of generators for the tensor product of the corresponding locales).

A.1.4. Push-outs of categories with finite limits. The (small, although this size condition will not play any role in the considerations here) categories with finite limits are, by definition, *algebraic theories*, and the left exact functors, *morphisms* of algebraic theories. They determine a 2-Category, which is in a sense like an algebraic category (the category of models of an algebraic theory). It is well known that it has all push-outs. They can be constructed, for example, as follows:

Given a diagram $\underline{H} \leftarrow \underline{E} \to \underline{G}$, consider the corresponding fibered category (over the category $I = \{\bullet \to \bullet \leftarrow \bullet\}$), and construct via a category of fractions its Limit (á 1a Grothendieck [12, VI, 5.5]). This Limit, that we denote $\underline{L}$, is a push-out of categories. Add (freely) all finite limits. For example, take the full subcategory $\underline{L}^\wedge$ of finitely presented objects in the functor category $(\mathbf{S}^{\underline{L}})^{op}$. Define a calculus of right fractions Σ in $\underline{L}^\wedge$ as follows: $s \in \Sigma \Leftrightarrow$ for all pairs of finite limit preserving functors

$h:\underline{H}\to\underline{X}$, $g:\underline{G}\to\underline{X}$ into a category $\underline{X}$ with finite limits, the corresponding functor $f:\underline{L}^{\wedge}\to X$ sends s into an invertible arrow in $\underline{X}$. Then the category of fractions $\underline{L}^{\wedge}[\Sigma^{-1}]$ is the desired push-out.

A.2. Morphisms and infinitesimal extension classifiers

A.2.1. Definition (Classifier of natural transformations between left exact functors).

Let $\underline{E}$ be any category with finite limits. A category with finite limits $\underline{M}$ together with a natural transformation between left exact functors $\mu:\partial_1^*\to\partial_0^*$, $\partial_0^*,\partial_1^*,:\underline{E}\to\underline{M}$, is the *generic natural transformation* if given any other such $\theta: p^*\to q^*, q^*, p^*:\underline{E}\to\underline{F}$, there exist $t^*:\underline{M}\to\underline{F}$ such that $t^*\mu=\theta$ (in the appropriate universal way).

A.2.2. Proposition. *In the situation of Definition* A.2.1, *assume there are topologies in* $\underline{E}$ *and* $\underline{M}$ *such that the topology in* $\underline{M}$ *is the coarsest which makes* ∂_0^* *and* ∂_1^* *continuous. Then the corresponding diagram between the topoi of sheaves is the morphism classifier* $\mathbf{E}^{\mathbf{M}}$ (*see Definition* 8.1).

Proof. To test whether $\mathrm{Sh}(\underline{M})=\mathbf{E}^{\mathbf{M}}$, consider a map between points $q,p:\mathbf{F}\to\mathbf{E},\theta:q\to p$, $\mathbf{E}=\mathrm{Sh}(E)$. Let $t^*:M\to\mathbf{F}$, $t^*\mu=\theta$, be the left exact functor whose existence is guaranteed by Definition A.2.1. Consider the topology in $\underline{M}$ whose covering families are those families which are sent by t^* into epimorphic families in $\mathbf{F}$. Clearly, by the assumption made, this topology contains the topology of the site $\underline{M}$. Thus t^* is continuous. Again, as in Proposition A.1.1, an immediate application of [11, SGA4 IV 4.9.4] finishes the proof. □

Corollary (existence of the exponential $\mathbf{E}^{\mathbf{M}}$). *Remark that the proposition above, in view of Proposition* A.2.5 *below, also serves as a proof for the existence of the exponential* $\mathbf{E}^{\mathbf{M}}$.

A.2.3. Proposition. *In the situation of Definition* A.2.1 *and Proposition A.2.2, consider a generating pretopology in* $\underline{E}$. *Then the class of families* $M_\alpha\to M$ *in* $\underline{M}$ *obtained by pulling-back pretopology covers as indicated below*:

$$(1)\quad \begin{tikzcd} M \arrow[r] & \partial_0^*(X) & X \\ M_\alpha \arrow[u] \arrow[r] & \partial_0^* X_\alpha \arrow[u] & X_\alpha \arrow[u] \end{tikzcd} \qquad (2)\quad \begin{tikzcd} M \arrow[r] & \partial_1^*(Y) & Y \\ M_\alpha \arrow[u] \arrow[r] & \partial_1^*(Y)_\alpha \arrow[u] & Y_\alpha \arrow[u] \end{tikzcd}$$

is stable by pull-backs, and if we close it by composition we obtain a pretopology that generates the topology in M.

Proof. It is straightforward. $\square$

A.2.4. Proposition. *In the situation above, consider a class of arrows $\underline{A}$ in $\underline{E}$ satisfying the first group of axioms for etal classes. Each arrow $h : X \to Y$ in $\underline{E}$ determines the following diagram in $\underline{M}$*

$$\begin{tikzcd}
\partial_1^*(X) \arrow[rr, "\partial_1^*(h)"] \arrow[dd, "\mu X"'] \arrow[dr, "\sigma_h"] & & \partial_1^*(Y) \arrow[dd, "\mu Y"] \\
& P \arrow[ur] & \\
\partial_0^*(X) \arrow[rr, "\partial_0^*(h)"] \arrow[ur] & & \partial_0^*(Y)
\end{tikzcd}$$

where the lower triangular square is a pull back. Consider the class of all arrows in $\underline{M}$ obtained by pulling back arrows of the form σ_h for h in $\underline{A}$, and close this class by composition. This defines a pretopology on $\underline{M}$ (where all covering families are singletons). Then, composites of these covers with the two types of covers considered in Proposition A.2.3 form a pretopology, and the corresponding diagram between the topoi of sheaves is the infinitesimal extension classifier $\mathbf{E}^{(\mathbf{M})}$ (as defined in Proposition 8.2), *with respect to the etal class* $\mathbf{A}$ *generated by $\underline{A}$ in the topos* $\mathbf{E} = \mathrm{Sh}(\underline{E})$.

Proof. By Proposition A.2.3 it is clear that this pretopology defines a subtopos of $\mathbf{E}^{\mathbf{M}}$, in which the outer square (3) above becomes a quasipullback for all arrows h in $\underline{A}$. Then it follows from Definition–Proposition 1.4.1 that it is actually a pull-back, and this is so for all arrows h in $\mathbf{A}$. This shows that this subtopos is the subtopos $\mathbf{E}^{(\mathbf{M})}$ defined in Proposition 8.2. $\square$

The classifier for natural transformations between left exact functors (in the sense of A.2.1) always exists, and this is all we need in this paper. The construction that we shall briefly indicate below in Proposition A.2.5a) and the explicit description of a site for the morphism and infinitesimal extension classifiers that follows Proposition A.2.5b), are due to Joyal (see [3, 7]).

A.2.5. Proposition (Joyal). *Let E be any category with finite limits. Then*

(a) $\quad \mu : \partial_1^* \to \partial_0^*, \quad \partial_0^*, \partial_1^*, : \underline{E} \to \underline{E}^{2op}.$

$\partial_1^*(X) = (id : X \to X)$, $\partial_0^*(X) = (\pi : X \to 1)$ (*μ the obvious natural transformation*) *is the* generic natural transformation *between left exact functors* (*as in Definition* A.2.1.).

(b) *The covers* (1)–(3) *in Proposition* A.2.3 *and* A.2.4 *of an arbitrary object $M = (f : X \to Y)$ in $\underline{M} = \underline{E}^{2op}$ are of the following form* (*for $X_\alpha \to X$, $Y_\alpha \to Y$ covers in*

$\underline{E}$, and h in $\underline{A}$):

$$
(1)\quad \begin{array}{ccc} X_\alpha & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y & \xrightarrow{id} & Y \end{array}
\qquad (2)\quad \begin{array}{ccc} P_\alpha & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y_\alpha & \longrightarrow & Y \end{array}
\qquad (3)\quad \begin{array}{ccc} X & \xrightarrow{id} & X \\ \downarrow & & \downarrow f \\ H & \xrightarrow{h} & Y \end{array}
$$

where the squares in (2) *are pull-backs, and* (3) *is any factorization of f with h in A.*

Proof. (a) Given any object $(f:X\to Y)$ in E^{2op}, it is immediate to check that the following is a pull-back diagram:

$$
(4)\quad \begin{array}{ccc} (f:X\longrightarrow Y) & \xrightarrow{(f,id)} & \partial_1^*(Y) \\ \downarrow (id,\pi) & & \downarrow \mu Y \\ \partial_0^*(X) & \xrightarrow{\partial_0^*(f)} & \partial_0^*(Y) \end{array}
$$

From this, the rest of the proof is straightforward.

(b) It follows easily by a calculation based on diagram (4) above. □

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